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Some Holomorphic Functions connected with the Collatz Problem

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1 Collatz Problem

In this paper, we consider some holomorphic functions connected with the Collatz problem. The Collatz problem (or conjecture) is well known under the name $3n+1$ problem:
Take any positive integer n. If n is even, replace it by $\frac{n}{2}$; if n is odd, replace it by $3n+1$. Show that after finitely many such steps, this process reaches the number 1.

We bring odd numbers into focus. For non-zero integers, we define a function ϕ as follows: For a non-zero even integer n, $\phi(n)$ is a unique odd integer m such that

$$n=2^k m \quad (\exists k=1, 2, \cdots),$$

for an odd integer n, $\phi(n)$ is a unique odd integer m such that

$$3n+1=2^k m \quad (\exists k=1, 2, \cdots).$$

Let OZ be the set of all odd integers. Then the above function maps

$$\phi: \mathbf{Z} \setminus \{0\} \quad \rightarrow \quad O\mathbf{Z}(\subset \mathbf{Z} \setminus \{0\}),$$

and we call the restriction (to $O\mathbf{Z}$)

$$\phi: O\mathbf{Z} \quad \rightarrow \quad O\mathbf{Z}$$

the *Collatz function*.
The Collatz problem asserts that, for each positive odd integer n, we have a positive integer k such that $\phi^k(n)=1$.

Now, we can easily compute the inverse image of the Collatz function $\phi: O\mathbf{Z} \rightarrow O\mathbf{Z}$.

Fact 1 For $\forall k \in \mathbf{Z}$, we have

$$\begin{cases} \phi^{-1}(6k+1)= \{4^n(8k+1)+4^{n-1}+\cdots+4+1 \in O\mathbf{Z} \mid n=0, 1, 2, \cdots\} \\ \phi^{-1}(6k+3)=\phi \\ \phi^{-1}(6k+5)= \{4^n(4k+3)+4^{n-1}+\cdots+4+1 \in O\mathbf{Z} \mid n=0, 1, 2, \cdots\} . \end{cases}$$

2 Some holomorphic functions on C

Here, we construct some holomorphic functions which agree on all positive odd integers with the Collatz function ϕ. We define a meromorphic function

$$G(z):=\textstyle\sum_{m=0}^{\infty}\phi(2m+1)\left(\frac{1}{(z-2m-1)^2}-\frac{1}{(2m+1)^2}\right) \text{ in } \mathbf{C},$$

and a holomorphic function

$$F(z):=\left(\frac{4}{\pi^2}\cos^2\frac{\pi z}{2}\right) G(z) \text{ on } \mathbf{C}.$$

From Fact 1, we see easily that, for all $z \in \mathbf{C}$,

$$G(z)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(6k+1)\left(\frac{1}{(z-(4^n(8k+1)+4^{n-1}+\cdots+4+1))^2}-\frac{1}{(4^n(8k+1)+4^{n-1}+\cdots+4+1)^2}\right)$$

$$+\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(6k+5)\left(\frac{1}{(z-(4^n(4k+3)+4^{n-1}+\cdots+4+1))^2}-\frac{1}{(4^n(4k+3)+4^{n-1}+\cdots+4+1)^2}\right).$$

This meromorphic function $G(z)$ has the Laurent expansion

$$G(z)=\frac{\phi(2n+1)}{(z-2n-1)^2}-\frac{\phi(2n+1)}{(2n+1)^2}+\sum_{n\neq m\geq 0}\phi(2m+1)\left(\frac{1}{4(m-n)^2}-\frac{1}{(2m+1)^2}\right)$$
$$+\sum_{k\geq 1}\frac{(k+1)}{2^{k+2}}\left(\sum_{n\neq m\geq 0}\frac{\phi(2m+1)}{(m-n)^{k+2}}\right)(z-2n-1)^k$$

around $2n+1$ in $\boldsymbol{C}$ ($n=0, 1, \cdots$), and

$$F(z)=\frac{2}{\pi^2}(1-\cos\pi(z-2n-1))G(z) \text{ on } \boldsymbol{C}.$$

Fact 2 For the entire function $F(z)$, we have the following:

(1) $F(2n+1)=\phi(2n+1)$ for each non-negative integer n, and
$F(2n+1)=0$ for each negative integer n.

(2) $F'(2n+1)=0$ for each integer n.

Now, we also define

$$K_p(z):=\frac{4^p}{\pi^{2p}}\cos^{2p}\frac{\pi z}{2}\sum_{n=0}^{\infty}\frac{\phi(2n+1)}{(z-2n-1)^{2p}} \quad \text{on } \boldsymbol{C} \quad (p=2, 3, \cdots)$$

and

$$L_p(z):=-\frac{4^p}{\pi^{2p+1}}\cos^{2p}\frac{\pi z}{2}\sin\pi z\sum_{n=0}^{\infty}\frac{\phi(2n+1)}{(z-2n-1)^{2p+1}} \quad \text{on } \boldsymbol{C} \quad (p=1, 2, \cdots).$$

Fact 3 The following identities hold:

(1) $K_p(2n+1)=L_p(2n+1)=L_1(2n+1)=\phi(2n+1)$ $(n=0, 1, 2,\cdots, p=2, 3, \cdots)$, and
$K_p(2n+1)=L_p(2n+1)=L_1(2n+1)=0$ $(n=-1, -2, \cdots, p=2, 3, \cdots)$.

(2) $K'_p(2n+1)=L'_p(2n+1)=L'_1(2n+1)=0$ $(n\in\boldsymbol{Z}, p=2, 3, \cdots)$.

And we have

$$\begin{cases}(\sin\pi z)F'(z)-\pi(\cos\pi z-1)F(z)=2\pi L_1(z) \\ (1+\cos\pi z)F''(z)+2\pi(\sin\pi z)F'(z)+(2-\cos\pi z)\pi^2F(z)=3\pi^2K_2(z)\end{cases} \quad (\forall z\in\boldsymbol{C}).$$

We note that the function $y(z)=\cos\pi z+1$ on $\boldsymbol{C}$ satisfies the differential equations

$$\begin{cases}(\sin\pi z)y'(z)-\pi(\cos\pi z-1)y(z)=0 \\ (1+\cos\pi z)y''(z)+2\pi(\sin\pi z)y'(z)+(2-\cos\pi z)\pi^2y(z)=0.\end{cases}$$

Further we have, for all $z\in\boldsymbol{C}$,

$$\begin{cases}(\sin\pi z)K'_p(z)-p\pi(\cos\pi z-1)K_p(z)=2p\pi L_p(z) & (p=2, 3, \cdots) \\ (\sin\pi z)L'_p(z)-\pi((p+1)\cos\pi z-p)L_p(z)=-\frac{(2p+1)\pi}{2}(\cos\pi z-1)K_{p+1}(z) & \\ & (p=1, 2, \cdots).\end{cases}$$

The function $y(z)=(\cos\pi z+1)^p$ on $\boldsymbol{C}$ satisfies the differential equations

$$\begin{cases}(\sin\pi z)y'(z)-p\pi(\cos\pi z-1)y(z)=0 \\ (\sin\pi z)y''(z)-\pi(p\cos\pi z-p+1)y'(z)=0\end{cases} \quad (p=1, 2, \cdots).$$

3 Attractive fixed points and the Fatou set

Here we state some propositions concerning the Fatou set and the Julia set of the entire function $F(z)$. First, for $\forall x<0$ in $\boldsymbol{R}$ we find that

$$0>G(x)=\sum_{m=0}^{\infty}\phi(4m+1)\left(\frac{1}{(x-4m-1)^2}-\frac{1}{(4m+1)^2}\right)+\sum_{m=0}^{\infty}\phi(4m+3)\left(\frac{1}{(x-4m-3)^2}-\frac{1}{(4m+3)^2}\right)$$

$$>\frac{1}{(1-x)^2}-1+\frac{5}{(3-x)^2}-\frac{5}{3^2}+\frac{1}{(5-x)^2}-\frac{1}{5^2}+\frac{11}{(7-x)^2}-\frac{11}{7^2}+\frac{7}{(9-x)^2}-\frac{7}{9^2}+\frac{17}{(11-x)^2}-\frac{17}{11^2}$$

$$-\frac{3}{16}\log\frac{(9-x)}{9}-\frac{7}{36}+\frac{7}{4(9-x)}-\frac{3}{8}\log\frac{(11-x)}{11}-\frac{17}{44}+\frac{17}{4(11-x)},$$

because

$$0>\sum_{m=3}^{\infty}\phi(4m+1)\left(\frac{1}{(x-4m-1)^2}-\frac{1}{(4m+1)^2}\right)+\sum_{m=3}^{\infty}\phi(4m+3)\left(\frac{1}{(x-4m-3)^2}-\frac{1}{(4m+3)^2}\right)$$

$$>\sum_{m=3}^{\infty}(3m+1)\left(\frac{1}{(x-4m-1)^2}-\frac{1}{4m+1)^2}\right)+\sum_{m=3}^{\infty}(6m+5)\left(\frac{1}{(x-4m-3)^2}-\frac{1}{(4m+3)^2}\right)$$

$$>\int_2^{\infty}\left(\frac{3t+1}{(x-4t-1)^2}-\frac{3t+1}{(4t+1)^2}\right)\mathrm{dt}+\int_2^{\infty}\left(\frac{6t+5}{(x-4t-3)^2}-\frac{6t+5}{(4t+3)^2}\right)\mathrm{dt}$$

$$=-\frac{3}{16}\log\frac{(9-x)}{9}-\frac{7}{36}+\frac{7}{4(9-x)}-\frac{3}{8}\log\frac{(11-x)}{11}-\frac{17}{44}+\frac{17}{4(11-x)}.$$

Fact 4 We have the following:

(1) $F(x)>0$ for $\forall x>0$ in $\boldsymbol{R}$.

(2)
$$0\geq F(x)\geq\frac{4}{\pi^2}\left(\cos^2\frac{\pi x}{2}\right)\left(\frac{1}{(1-x)^2}-1+\frac{5}{(3-x)^2}-\frac{5}{3^2}+\frac{1}{(5-x)^2}-\frac{1}{5^2}\right.$$
$$+\frac{11}{(7-x)^2}-\frac{11}{7^2}+\frac{7}{(9-x)^2}-\frac{7}{9^2}+\frac{17}{(11-x)^2}-\frac{17}{11^2}$$
$$\left.-\frac{3}{16}\log\frac{(9-x)}{9}-\frac{7}{36}+\frac{7}{4(9-x)}-\frac{3}{8}\log\frac{(11-x)}{11}-\frac{17}{44}+\frac{17}{4(11-x)}\right)$$

for $\forall x\leq 0$ in $\boldsymbol{R}$.

The Taylor expansion of $G(z)$ around 0 in $\boldsymbol{C}$ is

$$G(z)=\sum_{k\geq 1}(k+1)\left(\sum_{m\geq 0}\frac{\phi(2m+1)}{(2m+1)^{k+2}}\right)z^k$$
$$=2\left(\sum_{m\geq 0}\frac{\phi(2m+1)}{(2m+1)^3}\right)z+3\left(\sum_{m\geq 0}\frac{\phi(2m+1)}{(2m+1)^4}\right)z^2+\cdots,$$

and

$$F(z)=\frac{2}{\pi^2}(1+\cos\pi z)G(z)\quad\text{on } \boldsymbol{C}.$$

Proposition 1 *For the entire function $F(z)$, we have the following:*

(1) *$z=0$ is a repelling fixed point of $F(z)$ such that $1.024<F'(0)<1.07$, where*

$$F'(0)=\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{\phi(2n+1)}{(2n+1)^3}$$
$$=\frac{8}{\pi^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{6k+1}{(4^n(8k+1)+4^{n-1}+\cdots+4+1)^3}+\frac{8}{\pi^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{6k+5}{(4^n(4k+3)+4^{n-1}+\cdots+4+1)^3}$$

($=1.043\cdots$ *derived by computer.*)

(2) *$z=1$ is a superattractive (namely $F'(1)=0$) fixed point of $F(z)$.*

(3) *There exists an attractive fixed point z_0 ($\in\boldsymbol{R}$) of $F(z)$ such that $-\frac{1}{20}<z_0<0$.*

Further, around $2n+1$ in $\boldsymbol{C}$ ($n=0, 1, \cdots$) we have

$$F(z)=\phi(2n+1)$$
$$+2\phi(2n+1)\sum_{m\geq 1}(-1)^m\left(\frac{1}{(2m+1)!}+\frac{1}{(2n+1)^2\pi^2(2m)!}\right)\pi^{2m}(z-2n-1)^{2m}$$
$$+\frac{2}{\pi^2}(1-\cos\pi(z-2n-1))\sum_{n\neq m\geq 0}\phi(2m+1)\left(\frac{1}{(z-2m-1)^2}-\frac{1}{(2m+1)^2}\right).$$

From the above expression, for $\forall n=0, 1, 2, \cdots$ we can compute that

$$|F(z)-F(2n+1)|<6(2n+1)\pi^2(z-2n-1)^2 \quad \text{if } |z-2n-1|<\frac{1}{\pi}.$$

Proposition 2 *Every positive odd integer is in the Fatou set $F(F)$ of the entire function $F(z)$. Moreover, for $\forall n=0, 1, 2, \cdots$ we have*

$$\left\{z\in \boldsymbol{C}\,\middle|\,|z-2n-1|<\frac{1}{12\pi^2(2n+1)}\right\}\subset F(F).$$

Fact 5 From Fact 4 (2) we have
(1) $0\geq F(x)\geq x+1$ for $\forall x\leq -1$ in $\boldsymbol{R}$.
(2) The composite F^n of F satisfies
$0\geq F^n(x)\geq -1$ if $0\geq x\geq -n-1$ in $\boldsymbol{R}$ $(\forall n=1, 2, \cdots)$.

Proposition 3 *Every negative odd integer is in the Julia set $J(F)$ of the entire function $F(z)$. Moreover, we have*

$$J(F)\cap(-\infty, 0]=\cup_{n>0}F^{-n}(0)\cap(-\infty, 0].$$

By THEOREM 3.1 of [1], we have the following:

Proposition 4 *Every component of the Fatou set of $F(z)$ is simply connected.*

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